Conjugate fillings and Legendrian weaves

Tianjin University Geometry & Topology Seminar

Wenyuan Li

Northwestern

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- For a Legendrian knot/link in S^3, we can study exact Lagrangian surfaces in D^4 with boundary on the Legendrian knot/link, called Lagrangian fillings.

Front projection of Legendrian links

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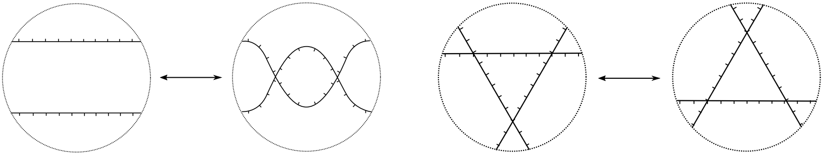

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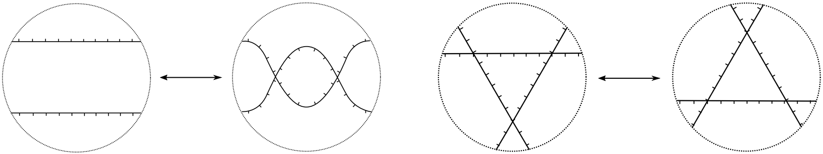
- Viewing D^4 as T^*D^2, when the Legendrian Λ is contained in S^*D^2, the projection onto D^2 is also called front projection. When moreover $\Lambda \subset S^*_{y_2<0}D^2_{x_1,x_2} \cong J^1D^1$, this recovers the standard front projection.



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3. Lagrangian projections of free Legendrian weaves (Treumann-Zaslow '16, Casals-Zaslow '20).

Recent results on Lagrangian fillings

- From 2012 to 16, different (but finite) Lagrangian fillings have been constructed using methods above and distinguished using algebraic invariants called **Legendrian contact homologies** (they are dg algebras defined by counting pseudo-holomorphic curves) or **constructible sheaves** (they are stratified local systems with respect to the front projection).

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3. Gao-Shen-Weng (Sep '20) showed infinite fillings using augmentation varieties (plus cluster algebra) for almost all positive braid closures;
4. Casals-Ng (Jan '21) showed infinite fillings using Legendrian contact homologies for another class of Legendrian links.

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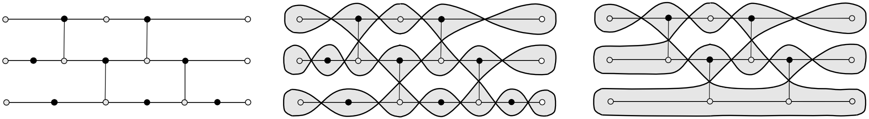
Theorem (Casals-L. 22')

For max-tb Legendrian positive braid closures (in fact a larger class), there is a Hamiltonian isotopy from (2) conjugate fillings of alternating Legendrians to certain (3) Lagrangian projections of Legendrian weaves.

Conjugate Lagrangian fillings

1. Start with a bicolored graph on the plane where all edges are between black and white vertices.



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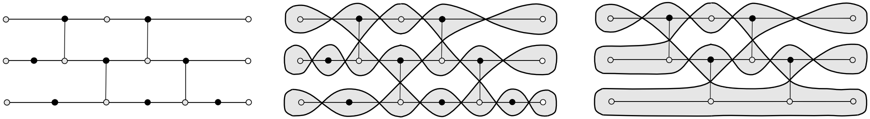
1. Start with a bicolored graph on the plane where all edges are between black and white vertices.
2. Consider a twisted ribbon of the graph with a twisting by π at every edge, called the conjugate surface.



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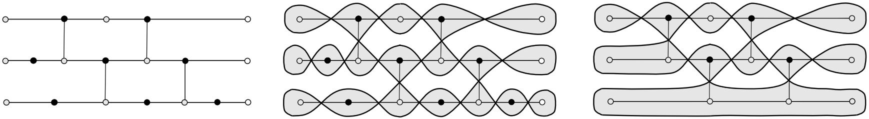
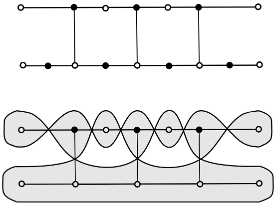
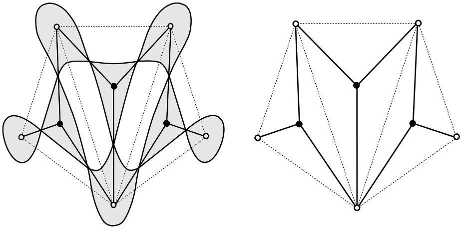
3. The boundary of the twisted ribbon is can be immersed into the plane, called the alternating strand diagram.



Conjugate Lagrangian fillings

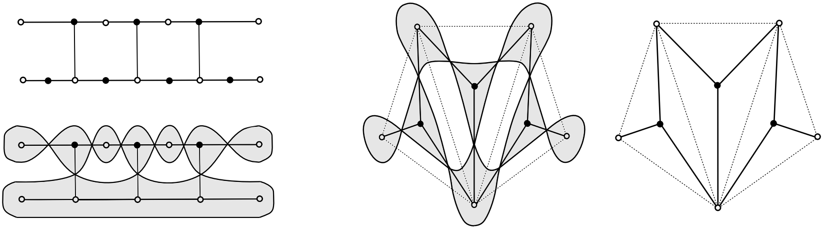
1. The alternating strand diagram lifts to the unit conormal bundle S^*D^2. This is called an alternating Legendrian.





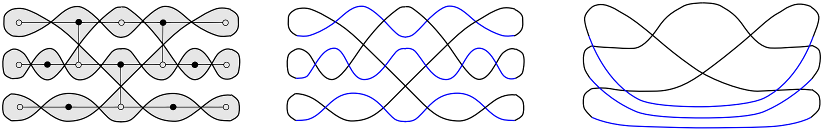
Conjugate Lagrangian fillings

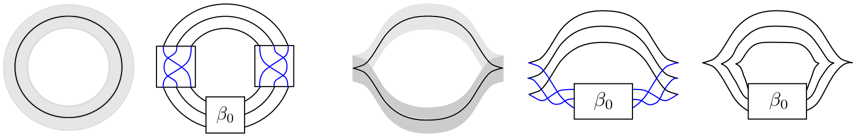
1. The alternating strand diagram lifts to the unit conormal bundle S^*D^2. This is called an alternating Legendrian.
2. The conjugate surface admits an exact Lagrangian embedding into T^*D^2, whose projection onto D^2 covers the black/white regions. This is the conjugate Lagrangian filling of the alternating Legendrian link (Shende-Treumann-Williams-Zaslow '15).



Conjugate Lagrangian fillings

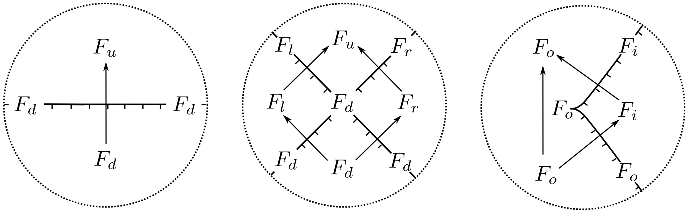
- For any Legendrian (rainbow) closure of positive braid, there is a bicolored graph on D^2 whose alternating Legendrian is isomorphic to the positive braid closure.





Conjugate Lagrangians and sheaves

- The construction of conjugate Lagrangians and alternating Legendrians allows one to apply the techniques of constructible sheaves easily.



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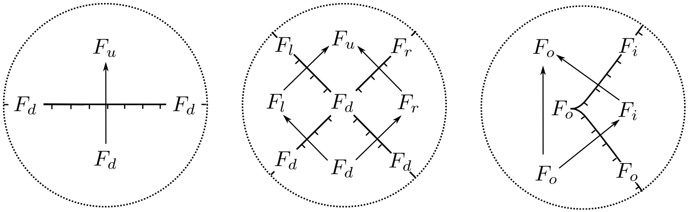
- The constructible sheaves we consider are sheaves on D^2 that are locally constant with respect to the stratification by the front projection $\pi(\Lambda)$ (with extra conditions). More precisely, they are sheaves with singular support on Λ.



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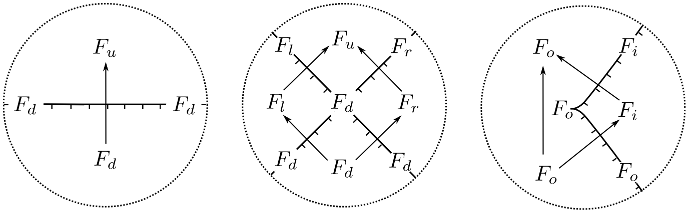
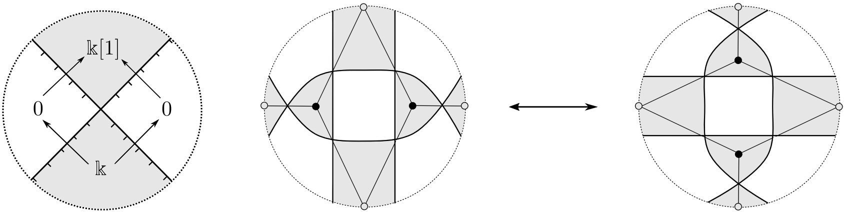
- We can consider microlocal rank 1 sheaves, which are constructible sheaves whose stalks jump by 1 when we cross the front projection.



Conjugate Lagrangians and sheaves

- Under some conditions, the moduli space of such sheaves forms a variety or a stack. A Lagrangian filling L gives rise to an open toric chart $H^1(L; \mathbb{C}^\times)$ parametrizing rank 1 local systems on L, which is an open subset of the stack.



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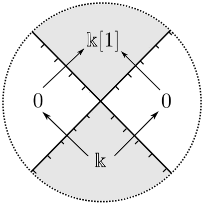
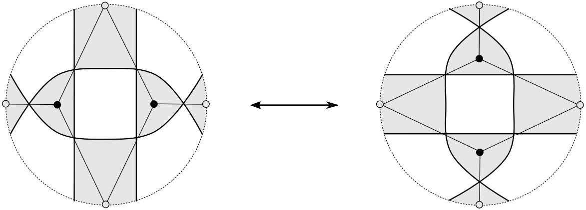
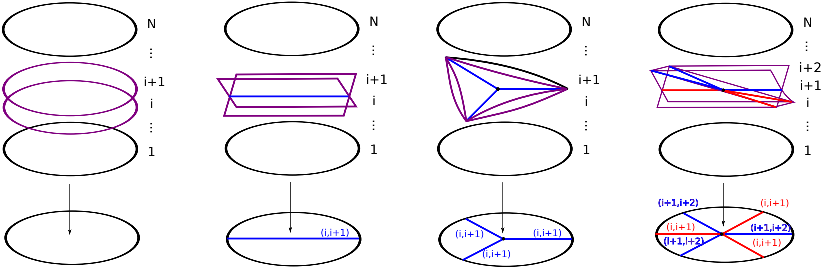
- For conjugate Lagrangian fillings, the corresponding sheaves are supported in the union of black and white regions with rank 1. Different conjugate Lagrangian fillings of the same Legendrian may give different toric charts.





- Legendrian weaves are Legendrian surfaces in $J^1\Sigma = T^*\Sigma \times \mathbb{R}$. Their projection onto Σ is a branched n-fold covering.



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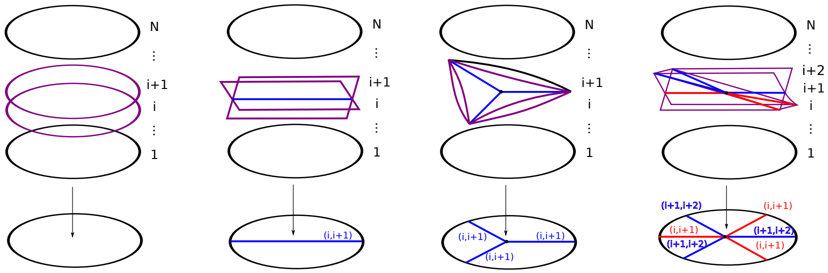
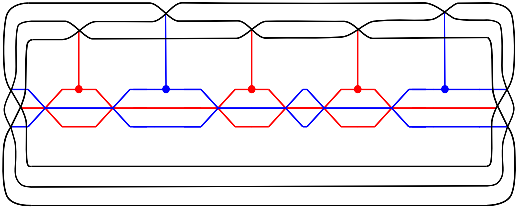
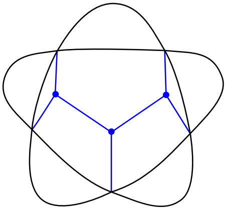
- The front projection of the Legendrian weave has n sheets of different heights at a generic points. Different sheets may intersect along some line segments. We encode the information of the front by drawing these lines of intersection.



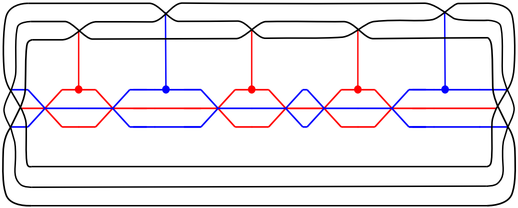
Legendrian weaves

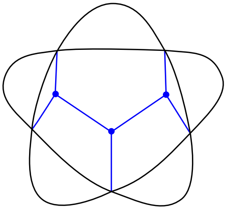
- For a Legendrian weave in $J^1 D^2$, the boundary is a Legendrian positive braid closure in $J^1 S^1$.



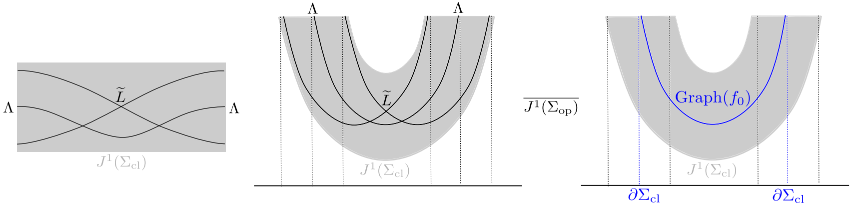


- For a Legendrian weave in J^1D^2, the boundary is a Legendrian positive braid closure in J^1S^1.
- The Lagrangian projection of the weave into T^*D^2 is an immersed Lagrangian filling of the Legendrian link in $T^*D^2|_{S^1} \cong J^1S^1$.





- Recall that as $D^4 \cong T^*D^2$, the boundary $S^3 = T^*D^2|_{S^1} \cup S^*D^2$. The Legendrian boundary of a weave lives in $T^*D^2|_{S^1}$ instead of S^*D^2.



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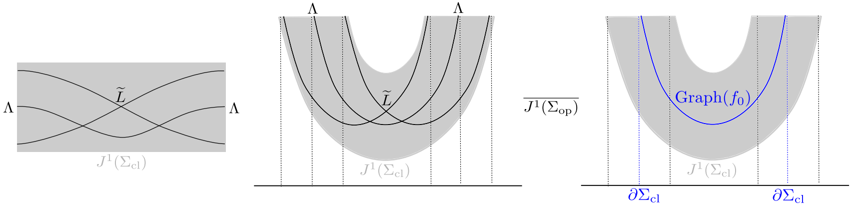
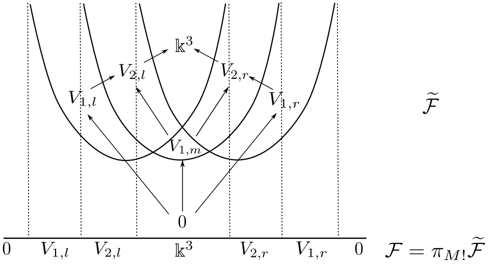
- We need to perturb the contact boundary into S^*D^2 in order to compare with conjugate Lagrangian fillings.



Legendrian weaves and sheaves

- The advantage of Legendrian weaves is that constructible sheaves with singular support on the weaves are easier to describe.



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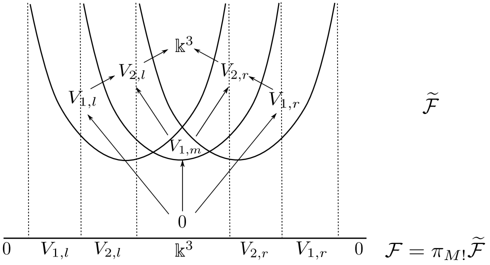
- Consider microlocal rank 1 sheaves. When there are n parallel planes in the front projection, the sheaf is determined by a flag $0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{k}^n$.



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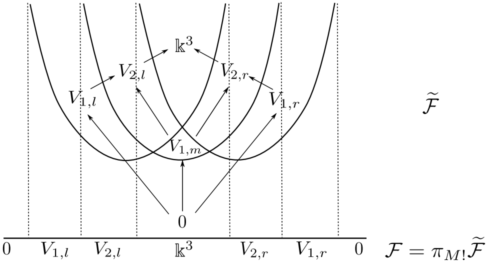
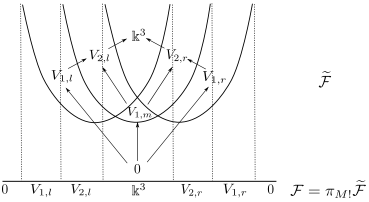
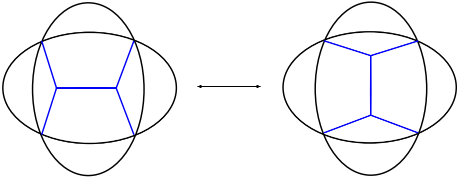
- When there are two transverse planes in the front projection, the corresponding flags satisfy a transverse condition.



Legendrian weaves and sheaves

- One can project the sheaf on $D^2 \times \mathbb{R}$ with singular support on the weave onto a sheaf on D^2 with singular support on the link.





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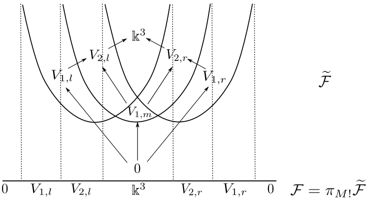

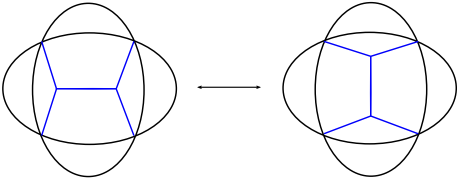

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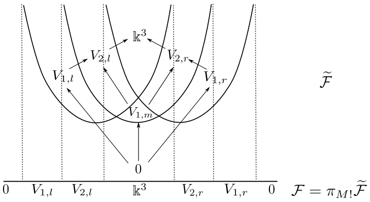
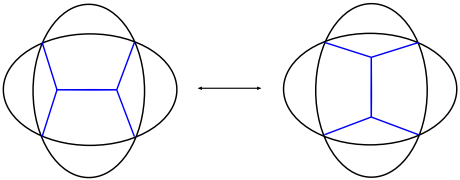
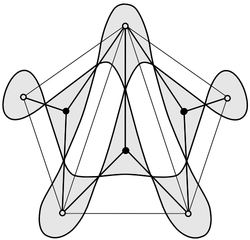
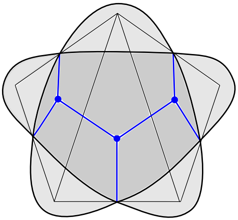




Comparison result

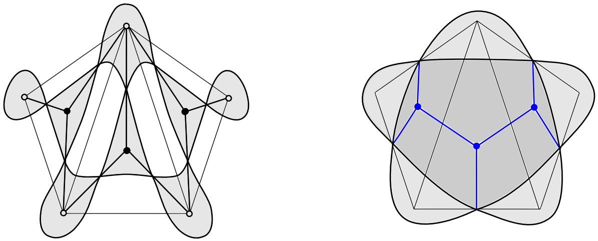
- First, we consider the simple case of Legendrian $(2, k)$-torus links (here we take $k = 3$).





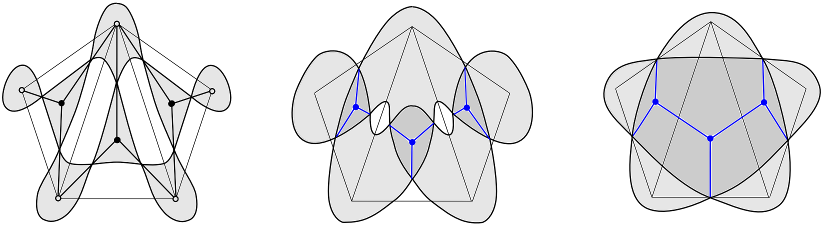
Comparison result

- First, we consider the simple case of Legendrian $(2, k)$-torus links (here we take $k = 3$).
- We claim that the following conjugate Lagrangian filling and Lagrangian projection of Legendrian weave are Hamiltonian isotopic.



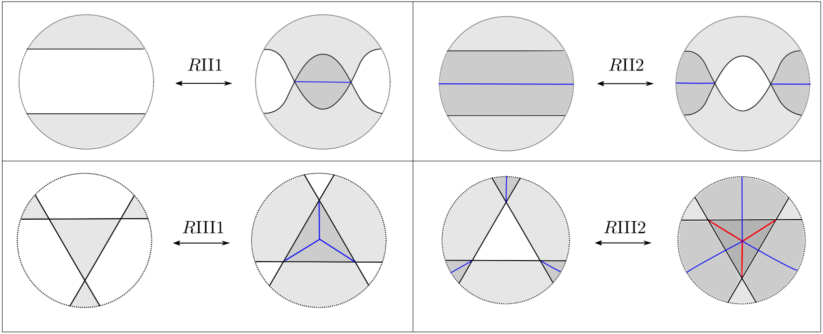
Comparison result

- We apply Reidemeister moves and to the Legendrian knot and guess the behaviour of the Lagrangian filling under these Reidemeister moves. If this is indeed the case, then the proof is completed.



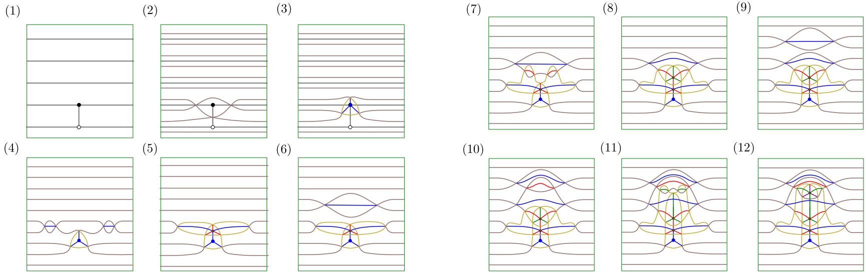
Comparison result

- We can summarize the local moves for Reidemeister moves as follows. This is the main technical result.



Comparison result

- Here is a more complicated example converts a more general conjugate filling to a Legendrian weave. This is the building block for the conjugate filling and Legendrian weave associated to a positive braid closure.



Algebraic implications

- Recall that constructible sheaves (of microlocal rank 1) with singular support on a Legendrian link Λ forms a variety or a stack, and a Lagrangian filling defines an open toric chart $H^1(L; \mathbb{C}^\times)$.

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- In many known cases, these moduli spaces are cluster varieties/stacks, meaning that it is covered (up to codimension 2) by toric charts, and the birational maps between toric charts are given by explicit combinatorial formulas.
- Conjugate fillings and Legendrian weaves give rise to different cluster coordinates. Our theorem implies that these coordinates are the same (due to Hamiltonian invariance of sheaf categories).

Thank you!